Deformations

• Recall: important *model* of shape change is the *deformation* of one form into another.
  – Dates back to D’Arcy Thompson’s (1917) transformation grids.
  – Deformation *maps* a set of morphological landmarks on one form onto the corresponding landmarks of another form.
    • Landmarks presumably homologous.
    • All other points are “carried along” (=interpolated) from one form to another.
    • *Elastic*: local compression/dilatation, no tears.
  – Deformation is a *smooth interpolation function*:
    • *Smooth* = 1\textsuperscript{st} and 2\textsuperscript{nd} derivatives exist (“doubly differentiable”).
    • Maps “known” landmarks onto corresponding landmarks exactly.
    • Predicts positions of points that lie among “known” landmarks: *mathematically homologous*. 
• **Problem**: there exists an infinite number of possible mappings between two landmark configurations.
  – Different mappings produce different interpolations.
• **Q**: How do we know which mapping is the ‘correct’ one?
  – **A**: We don’t.
  – Ontogenetic series: could potentially estimate local growth rates.
  – Static series: no comparable information available.
• A mapping is a model chosen for ‘nice’ (convenient) mathematical and statistical properties.
**Thin-plate spline (TPS)**

- Conventional mathematical tool for interpolating surfaces among scattered points in a 2D plane.
  - Generalization of the 1D cubic (polynomial) spline.
- Bookstein (1989) showed how TPS can be used to model and visualize deformations.
- Important property: **orthogonal decomposition**.
  - Can be decomposed into independent global and local components (analogous to regression):
    - Global component: *affine, uniform*
      - Single linear function.
      - Best approximates the overall deformation.
    - Local component: *non-affine, non-uniform*
      - Describes localized nonlinear residual adjustments.
      - Forces landmark positions in two forms to match exactly.
• **Affine** (uniform) transformations:
  - Act globally on all portions of the form.
  - Six affine transformations:
    • Four preserve shape: translation (A,B), scaling (C), rotation (D).
    • Two change shape: compression/dilatation in one direction (E), shearing (F).
Shift map

• *Shift map*: plot of the amount by which each landmark in the reference form must be shifted to map onto the corresponding landmark in the target form.

• Shift map in one dimension:
  – Deform the green creature so that its landmarks match those of the red creature.
• Shift map quantifies the shift for each landmark:

Landmark correspondences

Shifts

Shift amplitudes

Discontinuous shift function

Possible continuous shift function
• Cubic spline function:
  – Shift map is calculated as a weighted function.
  – Indicates *direction* and *magnitude* by which an arbitrary point must be shifted.
    • Point is weighted by its distances to all landmarks.
    • Weights: \(-U = -|d|^3\), where \(d\) is distance to landmark.

• 1D case is a “thin-wire” spline:
  – Analogous to bending a wire until landmarks line up.
  – *Bending energy* is proportional to curvature.
  – Bends so as to minimize bending energy of wire.
• Total deformation can be decomposed into:
  – *Linear global* (affine, uniform) component.
  – Residual *nonlinear local* (non-affine, non-uniform) component.
• Total deformation is sum of the two components.
• Two-dimensional spline:
  - Interpolated shifts in arbitrary points are in two dimensions, calculated separately.
  - Interpolated shifts are weighted sums of landmark shifts.
  - Weights: $-U = -d^2 \log d^2$ for distance $d$ from point to landmark.
  - Interpolation done for intersection points of an arbitrary grid.
• Shift maps in two dimensions:
• Separate shift maps for the $X$ and $Y$ dimensions:
• Interpolation of arbitrary point:
Examples of thin-plate splines for simple deformations:
• 2D splines are “thin-plate” splines:
  – Total deformation viewed as the projection into 2D of a uniformly thin metal plate.
    • Rotated and deformed in 3D.
    • Amount of local deformation measured as the bending energy required to deform such a plate.
    • Local bending energy proportional to curvature.
  – Thin-plate spline is that deformation that minimizes total bending energy (total curvature).

Opercular form in sticklebacks
• 2D thin-plate spline can be decomposed into *global* and *local* components:
  - *Affine component* is a tilted plane viewed in perspective.
  - *Non-affine component* characterizes regional deformations (warping of the thin plate).
  - *Total deformation* is sum of the two components.
• Ontogenetic change (allometry) in a piranha:

Total deformation

Affine and non-affine components
• Comparison of pelvis of *Australopithecus* and *Homo sapiens* (Berge, 1996):

Note: shouldn’t be straight lines in the total and non-affine spaces.
• Use of thin-plate splines to adjust facial profiles (Guillemaut et al. 2006):
• 3-dimensional “thin-hyperplate” spline:
  – Corresponds to distortion of a *malleable cube*.
  – Weights for interpolated shifts in arbitrary points are \(-U = -|d|\).
  – Stacked 2D planar grids are a sample of slices through a solid 3D grid.
    • Useful for visualization.
• Example of 3D thin-hyperplate spline:
  – Crania of *Caiman crocodilus* from Colombia.
  – 1 adult female, 1 adult male, 1 juvenile.
• Total deformation from juvenile to male.
  – Simulation of growth series.
  – Forms were superimposed by LSTRA.
  – Resulting deformation is function of allometric change plus small amount of individual variation.
• Total deformation from female to male.
  – Representation of *sexual dimorphism*.
  – Forms were superimposed by LSTRA.
  – Resulting deformation is function of sexual dimorphism plus small amount of individual variation.
Thin-plate splines – summary

- Corresponds to a 2D cubic-spline interpolation.
  - Convenient mathematical properties: smoothness, orthogonal decomposition.
  - Doesn’t necessarily correspond to meaningful biological interpolation.
- Typically begins with forms that have been standardized by Procrustes superimposition.
  - Isometric size adjustment via centroid size.
  - Allometric variation examined post hoc.
- Describes contrast between two forms: reference to target.
  - Can use consensus forms for groups of observations.
- Any statistical analysis that expresses results in terms of linear combinations of variables can be illustrated as a TPS.
  - Useful for examination of forms in tangent space.
- Non-affine component of the deformation can be further decomposed into orthogonal warps using eigenanalysis (later).